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Patterns of deformations of P_3 and P_4 breathers solutions to the NLS equation.

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Abstract

In this article, one gives a classification of the solutions to the one dimensional nonlinear focusing Schrödinger equation (NLS) by considering the modulus of the solutions in the (x, t) plan in the cases of orders 3 and 4. For this, we use a representation of solutions to NLS equation as a quotient of two determinants by an exponential depending on t . This formulation gives in the case of the order 3 and 4, solutions with respectively 4 and 6 parameters. With this method, beside Peregrine breathers, we construct all characteristic patterns for the modulus of solutions, like triangular configurations, ring and others.

1 Introduction

The rogue waves phenomenon currently exceed the strict framework of the study of ocean's waves [1, 2, 3, 4] and play a significant role in other fields; in nonlinear optics [5, 6], Bose-Einstein condensate [7], superfluid helium [8], atmosphere [9], plasmas [10], capillary phenomena [11] and even finance [12]. In the following, we consider the one dimensional focusing nonlinear equation of Schrödinger (NLS) to describe the phenomena of rogue waves. The first results concerning the NLS equation date from the Seventies. Precisely, in 1972 Zakharov and Shabat solved it using the inverse scattering method [13, 14]. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [15, 16]. The first quasi rational solutions of NLS equation were constructed in 1983 by Peregrine [17]. In 1986 Akhmediev, Eleonskii and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather [18, 19, 20]. Other analogues of the Peregrine breathers of order 3 and 4 were constructed in a series of articles by Akhmediev et al. [21, 22, 23] using Darboux transformations.

The present paper presents multi-parametric families of quasi rational solutions of NLS of order N in terms of determinants of order $2N$ dependent on $2N - 2$ real parameters.

The aim of this paper is to try to distinguish among all the possible configurations obtained by different choices of parameters, one those which have a characteristic in order to try to give a classification of these solutions.

2 Expression of solutions of NLS equation in terms of a ratio of two determinants

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0. \quad (1)$$

To solve this equation, we need to construct two types of functions $f_{j,k}$ and $g_{j,k}$ depending on many parameters. Because of the length of their expressions, one defines the functions $f_{\nu,\mu}$ and $g_{\nu,\mu}$ of argument A_ν and B_ν only in the appendix.

We have already constructed solutions of equation NLS in terms of determinants of order $2N$ which we call solution of order N depending on $2N - 2$ real parameters. It is given in the following result [24, 25, 26, 27] :

Theorem 2.1 *The functions v defined by*

$$v(x, t) = \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} e^{2it - i\varphi} \quad (2)$$

are quasi-rational solution of the NLS equation (1) depending on $2N - 2$ parameters $\tilde{a}_j, \tilde{b}_j, 1 \leq j \leq N - 1$, where

$$\begin{aligned} n_{j1} &= f_{j,1}(x, t, 0), \\ n_{jk} &= \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= f_{j,N+1}(x, t, 0), \\ n_{jN+k} &= \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= g_{j,1}(x, t, 0), \\ d_{jk} &= \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= g_{j,N+1}(x, t, 0), \\ d_{jN+k} &= \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ 2 \leq k \leq N, 1 \leq j \leq 2N \end{aligned} \quad (3)$$

The functions f and g are defined in (9), (10), (11), (12).

3 Patterns of quasi rational solutions to the NLS equation

The solutions v_N to NLS equation (2) of order N depending on $2N - 2$ parameters \tilde{a}_j, \tilde{b}_j (for $1 \leq j \leq N - 1$) has been already explicitly constructed and can be written as

$$v_N(x, t) = \frac{n(x, t)}{d(x, t)} \exp(2it) \\ = (1 - \alpha_N \frac{G_N(2x, 4t) + iH_N(2x, 4t)}{Q_N(2x, 4t)}) e^{2it}$$

with

$$G_N(X, T) = \sum_{k=0}^{N(N+1)} g_k(T) X^k, \\ H_N(X, T) = \sum_{k=0}^{N(N+1)} h_k(T) X^k, \\ Q_N(X, T) = \sum_{k=0}^{N(N+1)} q_k(T) X^k.$$

For order 3 these expressions can be found in [28]; in the case of order 4, they can be found in [29]. In the following, based on these analytic expressions, we give a classification of these solutions by means of patterns of their modulus in the plane $(x; t)$.

3.1 Patterns of quasi rational solutions of order 3 with 4 parameters

3.1.1 P_3 breather

If we choose all parameters equal to 0, $\tilde{a}_1 = \tilde{b}_1 = \dots = \tilde{a}_{N-1} = \tilde{b}_{N-1} = 0$, we obtain the classical Peregrine breather given by

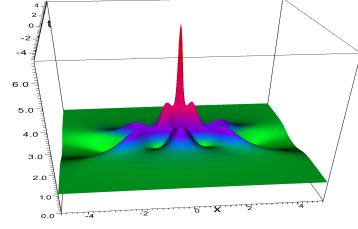


Figure 1: Solution P_3 to NLS, $N=3$, $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = 0$.

3.1.2 Triangles

To shorten, the following notations are used : for example the sequence 1A3 + 1T3 means that the structure has one arc of 3 peaks and one triangle of 3 peaks.

If we choose \tilde{a}_1 or \tilde{b}_1 not equal to 0 and all other parameters equal to 0, we obtain triangular configuration with 6 peaks.

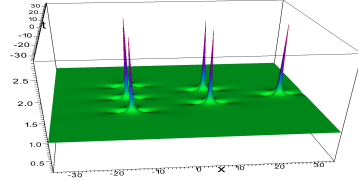


Figure 2: Solution 1T6 to NLS, $N=3$, $\tilde{a}_1 = 10^4, \tilde{b}_1 = 0, \tilde{a}_2 = 0, \tilde{b}_2 = 0$.

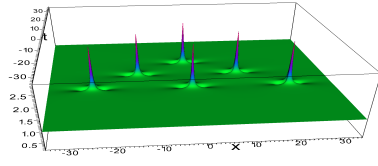


Figure 3: Solution 1T6 to NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^4$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$.

(and vice versa, \tilde{b}_1 and \tilde{b}_2 not equal to 0 and all other parameters equal to 0), we obtain deformed triangular configuration which can call arc structure.

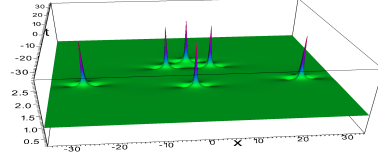


Figure 6: Solution 1A3 + 1T3 to NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^4$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 5 \times 10^6$.

3.1.3 Rings

If we choose \tilde{a}_2 or \tilde{b}_2 not equal to 0, all other parameters equal to 0, we obtain ring configuration with peaks.

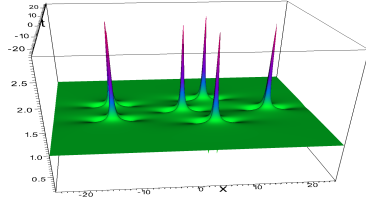


Figure 4: Solution 1R5 + 1 to NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$.

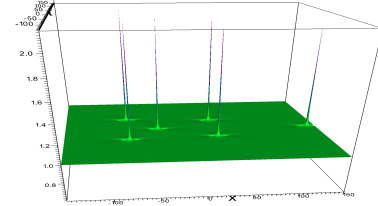


Figure 7: Solution 1A3 + 1T3 to NLS, $N=3$, $\tilde{a}_1 = 10^6$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^{10}$, $\tilde{b}_2 = 0$.

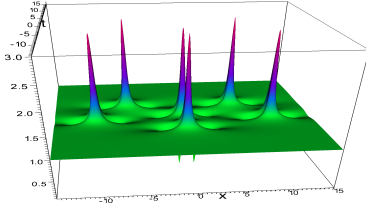


Figure 5: Solution 1R5 + 1 to NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^5$.

3.2 Patterns of quasi rational solutions of order 4 with 6 parameters

3.2.1 P_4 breather

If we choose all parameters equal to 0, $\tilde{a}_1 = \tilde{b}_1 = \dots = \tilde{a}_{N-1} = \tilde{b}_{N-1} = 0$, we obtain the classical Peregrine breather given in the following figure

3.1.4 Arcs

If we choose \tilde{a}_1 and \tilde{a}_2 not equal to 0 and all other parameters equal to 0

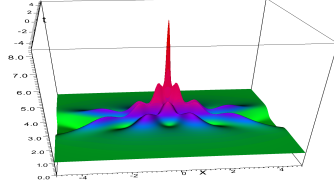


Figure 8: Solution P_4 to NLS, $N=4$, $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = \tilde{a}_3 = \tilde{b}_3 = 0$.

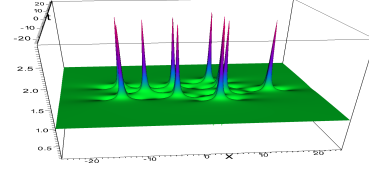


Figure 10: Solution $2R5/5$ to NLS, $N=4$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^5$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$.

3.2.2 Triangles

To shorten, we use the notations defined in the previous section.

If we choose \tilde{a}_1 or \tilde{b}_1 not equal to 0 and all other parameters equal to 0, we obtain triangular configuration with 10 peaks.

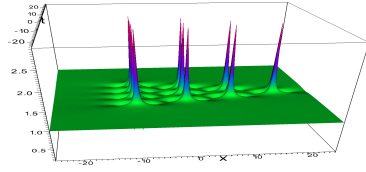


Figure 9: Solution $1T10$ to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$.

3.2.3 Rings

If we choose \tilde{a}_2 or \tilde{a}_3 not equal to 0 and all other parameters equal to 0 (or vice versa \tilde{b}_2 or \tilde{b}_3 not equal to 0 and all other parameters equal to 0), we obtain ring configuration with 10 peaks.

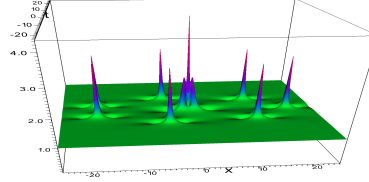


Figure 11: Solution $1R7 + P_2$ to NLS, $N=4$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 10^8$, $\tilde{b}_3 = 0$.

3.2.4 Arcs

If we choose two parameters non equal to 0, \tilde{a}_1 and \tilde{a}_2 , or \tilde{a}_1 and \tilde{a}_3 not equal to 0, or \tilde{a}_2 and \tilde{a}_3 and all other parameters equal to 0 (or vice versa for parameters b), we obtain arc configuration with 10 peaks¹.

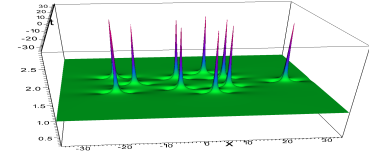


Figure 12: Solution $2A3/4I + T3$ to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$.

¹In the following notations $2A4/3I$, I meaning Reversed

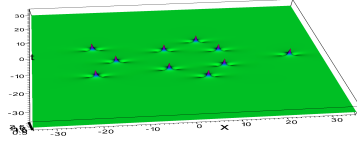


Figure 13: Solution $2A3/4I + T3$ to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 0$, $\tilde{b}_3 = 0$, sight top.

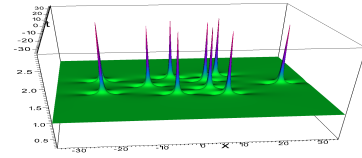


Figure 16: Solution $2A3/4 + 1T3$ to NLS, $N=4$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 3 \times 10^8$, $\tilde{b}_3 = 0$.

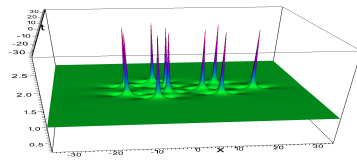


Figure 14: Solution $2A4/3 + 1T3$ to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 5 \times 10^7$, $\tilde{b}_3 = 0$.

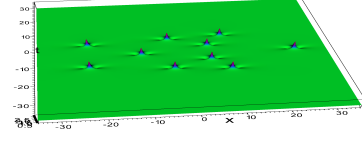


Figure 17: Solution $2A3/4 + 1T3$ to NLS, $N=4$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 3 \times 10^8$, $\tilde{b}_3 = 0$, sight top.

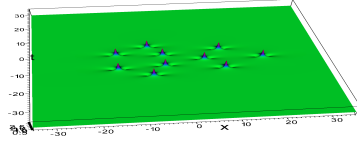


Figure 15: Solution $2A4/3 + 1T3$ to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 5 \times 10^7$, $\tilde{b}_3 = 0$, sight top.

3.2.5 Triangles inside rings

If we choose three parameters non equal to 0, \tilde{a}_1 , \tilde{a}_2 and \tilde{a}_3 and all other parameters equal to 0 (or vice versa for parameters b), we obtain ring with inside triangle.

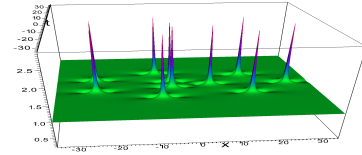


Figure 18: Solution $1A7 + 1T3$ to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^3$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 10^9$, $\tilde{b}_3 = 0$.

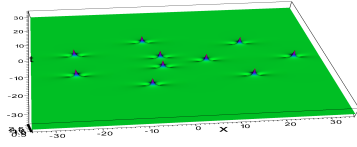


Figure 19: Solution 1A7+1T3 to NLS, $N=4$, $\tilde{a}_1 = 10^3$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^3$, $\tilde{b}_2 = 0$, $\tilde{a}_3 = 10^9$, $\tilde{b}_3 = 0$, sight top.

4 Conclusion

We recall one more time that the solutions at order 3 and 4 to the equation NLS dependent on 4 and 6 parameters were given for the first time by V.B. Matveev [49]. The solutions and their deformations presented here by the authors were built later by a completely different method [28], [29].

We have presented here patterns of modulus of solutions to the NLS focusing equation in the (x, t) plane.

These study can be useful at the same time for hydrodynamics as well for non-linear optics; many applications in these fields have been realized, as it can be seen in recent works of Chabchoub et al. [50] or Kibler et al. [51].

This study try to bring all possible types of patterns of quasi rational solutions to the NLS equation.

We see that we can obtain 2^{N-1} different structures at the order N .

Parameters a or b give the same type of structure. For $a_1 \neq 0$ (and other parameters equal to 0), we obtain triangular rogue wave; for $a_j \neq 0$ ($j \neq 1$ and other parameters equal to 0) we get ring rogue wave; in the other choices of parameters, we get in particular arc structures (or claw structure).

This type of study have been realized

in preceding works. Akhmediev et al study the order $N = 2$ in [52], $N = 3$ in [53]; the case $N = 4$ was studied in particular ($N = 5, 6$ were also studied) in [54, 55] showing triangle and arc patterns; only one type of ring was presented. The extrapolation was done until the order $N = 9$ in [56]. Ohta and Yang [57] presented the study of the case cas $N = 3$ with rings and triangles. Recently, Ling and Zhao [58] presented the cases $N = 2, 3, 4$ with rings, triangle and also claw structures. In the present study, one sees appearing richer structures, in particular the appearance of a triangle of 3 peaks inside a ring of 7 peaks in the case of order $N = 4$; to the best of my knowledge, it is the first time that this configuration for order 4 is presented.

In this way, we try to bring a better understanding to the hierarchy of NLS rogue wave solutions.

It will be relevant to go on this study with higher orders.

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Appendix

Parameters and functions

We consider the terms λ_ν satisfying the relations for $1 \leq j \leq N$

$$\begin{aligned} 0 < \lambda_j < 1, \quad \lambda_{N+j} &= -\lambda_j, \\ \lambda_j &= 1 - 2\epsilon^2 j^2, \end{aligned}$$

with ϵ a small number intended to tend towards 0.

The terms κ_ν , δ_ν , γ_ν are functions of the parameters λ_ν , $1 \leq \nu \leq 2N$. They are given by the following equations, for $1 \leq j \leq N$:

$$\begin{aligned} \kappa_j &= 2\sqrt{1 - \lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \\ \gamma_j &= \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}, \end{aligned} \quad (4)$$

$$\begin{aligned} \kappa_{N+j} &= \kappa_j, \quad \delta_{N+j} = -\delta_j, \\ \gamma_{N+j} &= 1/\gamma_j. \end{aligned} \quad (5)$$

The terms $x_{r,\nu}$ $r = 3, 1$ are defined by

$$\begin{aligned} x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \\ x_{r,N+j} &= (r-1) \ln \frac{\gamma_{N+j} - i}{\gamma_{N+j} + i}. \end{aligned} \quad (6)$$

The parameters e_ν are given by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad (7)$$

where a_j and b_j are chosen in the form

$$\begin{aligned} a_j &= \sum_{k=1}^{N-1} \tilde{a}_k \epsilon^{2k+1} j^{2k+1}, \\ b_j &= \sum_{k=1}^{N-1} \tilde{b}_k \epsilon^{2k+1} j^{2k+1}, \\ 1 &\leq j \leq N, \end{aligned} \quad (8)$$

with \tilde{a}_j , \tilde{b}_j , $1 \leq j \leq N-1$, $2N-2$, arbitrary real numbers.

The functions $f_{\nu,1}$ and $g_{\nu,1}$, $1 \leq \nu \leq N$ are defined by (here $k = 1$) :

$$\begin{aligned} f_{4j+1,1} &= \gamma_k^{4j-1} \sin A_1, \\ f_{4j+2,1} &= \gamma_k^{4j} \cos A_1, \\ f_{4j+3,1} &= -\gamma_k^{4j+1} \sin A_1, \\ f_{4j+4,1} &= -\gamma_k^{4j+2} \cos A_1, \end{aligned} \quad (9)$$

$$\begin{aligned} f_{4j+1,N+1} &= \gamma_k^{2N-4j-2} \cos A_{N+1}, \\ f_{4j+2,N+1} &= -\gamma_k^{2N-4j-3} \sin A_{N+1}, \\ f_{4j+3,N+1} &= -\gamma_k^{2N-4j-4} \cos A_{N+1}, \\ f_{4j+4,N+1} &= \gamma_k^{2N-4j-5} \sin A_{N+1}, \end{aligned} \quad (10)$$

$$\begin{aligned} g_{4j+1,1} &= \gamma_k^{4j-1} \sin B_1, \\ g_{4j+2,1} &= \gamma_k^{4j} \cos B_1, \\ g_{4j+3,1} &= -\gamma_k^{4j+1} \sin B_1, \\ g_{4j+4,1} &= -\gamma_k^{4j+2} \cos B_1, \end{aligned} \quad (11)$$

$$\begin{aligned} g_{4j+1,N+1} &= \gamma_k^{2N-4j-2} \cos B_{N+1}, \\ g_{4j+2,N+1} &= -\gamma_k^{2N-4j-3} \sin B_{N+1}, \\ g_{4j+3,N+1} &= -\gamma_k^{2N-4j-4} \cos B_{N+1}, \\ g_{4j+4,N+1} &= \gamma_k^{2N-4j-5} \sin B_{N+1}. \end{aligned} \quad (12)$$

The arguments A_ν and B_ν of these functions are defined by $1 \leq \nu \leq 2N$:

$$\begin{aligned} A_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ B_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2. \end{aligned}$$